Bott. Cosimplicial construction of BG

(Bott's Harvard lecture, 1990 Fall) 2018 Fall.

§ First Stiefel-Whitney class.

Theorem. $\{\mathbb{R}\text{-line bundles}/M\}/\text{isom.} \simeq H^1(M,\mathbb{Z}_2)$

1st. Pf: line bdl. $\sim$ double cover

$\sim$ index 2 subgp. of $\pi_1(M)$

$\pi_1 \twoheadrightarrow \mathbb{Z}_2$

$w_1 \in \mathrm{Hom}\left(\frac{\pi_1}{[\pi_1,\pi_1]}, \mathbb{Z}_2\right) \simeq H^1(M,\mathbb{Z}_2)$

2nd. Pf. universal (most twisted) double cover

$n \gg 0$, $S^n \xrightarrow{\mathbb{Z}_2} \mathbb{RP}^n$, $\left(\mathcal{O}(1) \xrightarrow[\text{tauto. bdl}]{\mathbb{R}} \mathbb{RP}^n\right)$

$H^\bullet(\mathbb{RP}^n) \simeq \mathbb{Z}_2[x]/x^{n+1}$

Given $\mathbb{R} \longrightarrow L \longrightarrow M$,

find $f: M \longrightarrow \mathbb{RP}^n$ s.t. $L \simeq f^*\mathcal{O}(1)$

Step 1°. Find finite dim. $V \leqslant \Gamma(M,L)$

s.t. $\forall\ p \in M$, $ev_p : V \twoheadrightarrow L_p$

(i.e. globally generated)

2° $f_L: M \longrightarrow Gr(N-1,N) \simeq \mathbb{RP}^N$

$f_L(p) := \mathrm{Ker}(ev_p)$

Same for higher rank bdl, use $Gr(\overset{\text{codim}}{r}, N)$.

Theorem. $\{\text{rk } r\ VB/M\}/\text{isom} \simeq [M, B]$

$$B = \varinjlim_{N\to\infty} Gr(r,N) = Gr(r,\infty)$$

So, every $\omega \in H^\bullet(B)$ defines a functorial cohomology class for vector bundles.

$E/M \longmapsto f_E^*\omega \in H^\bullet(M)$

$H^\bullet(\mathbb{RP}^\infty, \mathbb{Z}_2) = \mathbb{Z}_2[x]$. $\quad x \rightsquigarrow$ 1st Stiefel-Whitney class.

2nd Pf. continued.

$$\{\text{line bdl.}/M\}/\text{isom.}$$

$$\simeq [M, \underbrace{\mathbb{RP}^\infty}_{K(\mathbb{Z}_2,1)}] \swarrow \begin{cases} \pi_1(\mathbb{RP}^\infty) = \mathbb{Z}_2 \\ \pi_{\neq 1}(-''-) = 0 \end{cases}$$

$$= H^1(M, \mathbb{Z}_2)$$

Eilenberg-Maclane space $K(\pi, n)$

Def$^{\underline{n}}$: $\begin{cases} \pi_n(K(\pi, n)) = \pi \\ \pi_{\neq n}(-''-) = 0 \end{cases}$

Property: $H^n(M, \pi) = [M, K(\pi, n)]$

Construct $K(\pi, n)$ via "cosimplicial" constr. s.t. $H^\bullet$ ✓

§ Cosimplicial construction, motivation

Recall $H^\bullet_{\text{sing}}(M, A) := H^\bullet(C^\bullet(M, A), \delta)$

$\qquad A$: Abelian group $\qquad C^\bullet$: {singular cochains}

$$C^n(M, A) \ni c : \quad \forall \ \varphi \xmapsto{\ c\ } c(\varphi) \in A$$

$$\varphi : \Delta_n \longrightarrow M$$

$\Delta_n \subset \mathbb{R}^{n+1}$ std. simplex.

- $\varphi : \Delta_n \longrightarrow X$

$\rightsquigarrow \delta\varphi : \Delta_{n+1} \longrightarrow X \quad$ s.t. $\delta^2 = 0$

w/ $(\delta\varphi)(x_0, \dots, x_n) := \sum_{j=0}^{n} (-1)^j \varphi(\underbrace{x_0, \dots, \hat{x}_j, \dots, x_n}_{})$

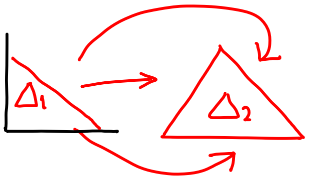
need normalize s.t. sum=0

$$\begin{tikzcd} C^0(M,A) \arrow[r, shift left] \arrow[r, shift right] & C^1(M,A) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & C^2(M,A) \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & C^3(M,A) \arrow[r, "\vdots"] & \cdots \end{tikzcd}$$

$\rightsquigarrow$ alternating

$$\xrightarrow{\ \delta\ } \qquad \xrightarrow{\ \delta\ } \qquad \xrightarrow{\ \delta\ } \qquad \xrightarrow{\ \delta\ } \ \cdots$$

$$\Delta_{n-1} \;\substack{\longrightarrow \\ \longrightarrow \\ \vdots\, n+1 \\ \longrightarrow}\; \Delta_n$$



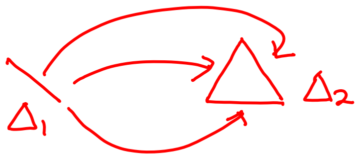
$$\Delta_2 = \{ (a_0, a_1, a_2) \in \mathbb{R}^3_{\geqslant 0} : \sum_{i=0}^{2} a_i = 1 \}$$

$$= \{ \text{probability measures on } \{n\} \}$$

$$\Delta : \overbrace{((\mathrm{Ord}))}^{\text{cat. of ordered finite set}} \longrightarrow ((\mathrm{Top})) \quad \text{covariant functor.}$$

$$[n] \longmapsto \Delta[n] = \Delta_n$$

$$[1] \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\; [2] \longmapsto$$



$$\uparrow$$

$$\left(\begin{array}{l} \{0,1\} \longrightarrow \{\cancel{0},1,2\} \\ \{0,1\} \longrightarrow \{0,\cancel{1},2\} \\ \{0,1\} \longrightarrow \{0,1,\cancel{2}\} \end{array} \right)$$

Def^{n}:

$$\left(\begin{array}{ll} \text{simplicial space}: & ((\mathrm{ord})) \longrightarrow ((\mathrm{Top})) \quad \text{covariant functor} \\ \text{cosimplicial space}: & ((\mathrm{ord})) \longrightarrow ((\mathrm{Top})) \text{ contravariant functor} \end{array} \right)$$

M topological space, i.e. $M \in \mathrm{Ob}((\mathrm{Top}))$

$$Y \longmapsto \mathrm{Hom}(Y, M) = \mathrm{Map}_{cts}(Y, M)$$

$$\begin{tikzcd} \rightsquigarrow ((\mathrm{Top})) \arrow[r, "\underline{M}"] & ((\mathrm{Set})) \\ ((\mathrm{Ord})) \arrow[u, "\Delta"] \arrow[ur, "S^{\underline{M}}"'] & \end{tikzcd}$$

contravariant functor

$$S^{\underline{M}}([n]) =: S^{\underline{M}}_n = \mathrm{Map}(\Delta_n, M)$$

(from $\mathrm{Hom}(\{n-1\}, \{n\})$)

$$S^{\underline{M}}_0 \;\substack{\longleftarrow \\ \longleftarrow}\; S^{\underline{M}}_1 \;\substack{\longleftarrow \\ \longleftarrow \\ \longleftarrow}\; S^{\underline{M}}_2 \;\substack{\longleftarrow \\ \longleftarrow \\ \longleftarrow \\ \longleftarrow}\; S^{\underline{M}}_3 \;\cdots\cdots$$

$\forall$ A Abelian group (coeff) ⇝

($\mathrm{Hom}(B,A)$ = A-mod. freely generated by B).

$$\begin{tikzcd}
((\mathrm{Top})) \arrow[r, "M"] & ((\mathrm{Set})) \arrow[r, "\mathrm{Hom}(-,A)"] & ((A\text{-mod})) \\
((\mathrm{Ord})) \arrow[u, "\Delta"] \arrow[ur, "S^M"] \arrow[urr, "C^\bullet(M,A)"'] & &
\end{tikzcd}$$

$$C^n(M,A) = C^\bullet(M,A)([n]) \ni \sum a_i \varphi_i \,, \quad \begin{array}{l}\varphi_i : \Delta_n \to X \\ a_i \in A\end{array}$$

$$\rightsquigarrow \quad C^0(X,A) \rightrightarrows C^1(X,A) \Rrightarrow C^2(X,A) \overset{\to}{\underset{\to}{\vdots}} \cdots\cdots$$

$$\begin{array}{ccc} \wr \text{ alternating} & \wr & \wr \\ \xrightarrow[\delta]{} & \xrightarrow[\delta]{} & \xrightarrow[\delta]{} \end{array}$$

$$\rightsquigarrow \quad H^\bullet_{sing}(X,A) := H^\bullet(C^\bullet(X,A), \delta).$$

§ Cosimplicial construction

Instead of given topo. space M ← recover by realizat^n

⇝ $S^M : ((\mathrm{ord})) \longrightarrow ((\mathrm{Top}))$

w/ $S^M([n]) =: S^M_n = \mathrm{Map}(\Delta_n, M)$

i.e. $S^M_0 \leftleftarrows S^M_1 \Lleftarrow S^M_2 \substack{\leftarrow\\ \leftarrow\\ \leftarrow\\ \leftarrow} S^M_3 \cdots\cdots$

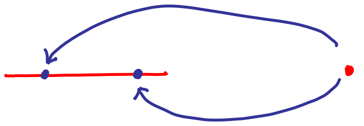
ANY $Y : ((\mathrm{ord})) \longrightarrow ((\mathrm{Top}))$ (cosimplicial set)

i.e. $Y_0 \leftleftarrows Y_1 \Lleftarrow Y_2 \substack{\leftarrow\\ \leftarrow\\ \leftarrow\\ \leftarrow} Y_3 \cdots\cdots$

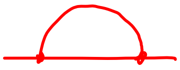
⇝ Realization $|Y| := \coprod_n Y_n \times \Delta_n / \sim$

$$\left(\begin{array}{l} \alpha \in \mathrm{Hom}([n-1],[n]) \rightsquigarrow Y(\alpha) : Y_n \longrightarrow Y_{n-1} \\ Y_{n-1} \times \Delta_{n-1} \ni (Y(\alpha)(y), x) \sim (y, \alpha(x)) \in Y_n \times \Delta_n \end{array} \right)$$

Eg. Y_0



Y_1 ⇒



Theorem (1) $\forall$ mfd. M, $|S^M| \xrightarrow{\sim} X$ weakly homotopy eq.

(2) $\forall$ simplicial complex K, $|S^K| \overset{h.e.}{\sim} K$

§ Construction of $K(\pi,1)$

- ∀ set Y ⇝ 2 cosimplicial sets, & map betʷ them:

all identity maps

$$\begin{tikzcd}
Y \arrow[d, "\Delta=1"'] & Y \arrow[l, shift left] \arrow[l, shift right] \arrow[d, "\Delta"'] & Y \arrow[l] \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[d, "\Delta"'] & \cdots \arrow[l, shift left=2] \arrow[l, phantom, "\vdots"] \arrow[l, shift right=2] \arrow[r, squiggly, "1\text{-}1"] & Y & (Ex) \\
Y & Y^2 \arrow[l, shift left] \arrow[l, shift right] & Y^3 \arrow[l] \arrow[l, shift left=2] \arrow[l, shift right=2] & \cdots \arrow[l, shift left=2] \arrow[l, phantom, "\vdots"] \arrow[l, shift right=2] \arrow[r, squiggly, "\text{realization}", "1\text{-}1"'] & * & (Ex)
\end{tikzcd}$$

$$\begin{tikzcd}
a & & (b,c) & \\
 & (a,b) \arrow[ul] \arrow[dl] & (a,c) & (a,b,c) \arrow[l] \arrow[ul] \arrow[dl] \\
b & & (a,b) &
\end{tikzcd}$$

When $Y = \pi$ is a group ⇝ quotient

$$\begin{tikzcd}
\pi \arrow[d] & Y \arrow[d, "\Delta=1"'] & Y \arrow[l, shift left] \arrow[l, shift right] \arrow[d, "\Delta"'] & Y \arrow[l] \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[d, "\Delta"'] & \cdots \arrow[l, shift left=2] \arrow[l, phantom, "\vdots"] \arrow[l, shift right=2] \\
E\pi \arrow[d] & Y \arrow[d] & Y^2 \arrow[l, shift left] \arrow[l, shift right] \arrow[d, "\text{quotient}"] & Y^3 \arrow[l] \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[d] & \cdots \arrow[l, shift left=2] \arrow[l, phantom, "\vdots"] \arrow[l, shift right=2] \\
B\pi & * & Y \arrow[l, shift left] \arrow[l, shift right] & Y^2 \arrow[l] \arrow[l, shift left=2] \arrow[l, shift right=2] & \cdots \arrow[l, shift left=2] \arrow[l, phantom, "\vdots"] \arrow[l, shift right=2]
\end{tikzcd}$$

$$\left(\begin{array}{ll} Y^4/Y & (a,b,c,d) \\ \wr\| & \\ Y^3 & (ba^{-1}, cb^{-1}, dc^{-1}) \end{array} \right.$$

$$\begin{tikzcd}
ba & & (b,c) & \\
b & (a,b) \arrow[l] \arrow[ul] \arrow[dl] & (ab,c) & (a,b,c) \arrow[l] \arrow[ul] \arrow[dl] \arrow[ddl] \\
a & & (a,bc) & \\
 & & (a,b) &
\end{tikzcd}$$

When π discrete group, $|E\pi| \sim *$, so

$$|B\pi| = K(\pi,1)$$

Note: Curv. of $M \leq 0 \Rightarrow M = K(\pi,1)$ w/ $\pi = \pi_1(M)$

In particular, $H^{\bullet}(M,\mathbb{Z}) = H^{\bullet}(\pi,\mathbb{Z})$.

$\forall$ Abelian group A, $H^\bullet(\pi, A) = H^\bullet(B\pi, A)$

Theorem.

$$H^0(\pi, A) = A$$
$$H^1(\pi, A) = \mathrm{Hom}(\pi, A)$$
$$H^2(\pi, A) = \{\text{central ext}^{\underline{n}} \text{ of } \pi \text{ by } A\}.$$

reason:

$$B\pi: \quad \cdot \Longleftarrow \pi \Lleftarrow \pi\times\pi \Lleftarrow \pi\times\pi\times\pi \quad \cdots\cdots$$

$$a, b, ab \Longleftarrow (a, b)$$

$$(b,c), (ab,c), (a,b), (a,bc) \Longleftarrow (a, b, c)$$

Apply $A \rightsquigarrow \underset{\rotatebox{90}{$\in$}}{A(\cdot)} \xrightarrow{\delta} A(\pi) \xrightarrow{\delta} A(\pi\times\pi) \xrightarrow{\delta} A(\pi\times\pi\times\pi) \cdots\cdots$

H^0: $\quad f: \bullet \longrightarrow A \rightsquigarrow \delta f = f(\cdot) - f(\cdot) = 0 \Rightarrow H^0 = A$

H^1: $\quad A(\pi) \ni f: \pi \longrightarrow A$

$$(\delta f)(a,b) = f(a) - f(ab) + f(b)$$

$$\Longrightarrow H^1 = \mathrm{Hom}(\pi, A)$$

H^2: $\quad A(\pi\times\pi) \ni f: \pi\times\pi \longrightarrow A$

$$0 = (\delta f)(a,b,c) = f(b,c) - f(ab,c) + f(a,bc) - f(a,b)$$

using multiplicative convention, i.e.

$$f(ab, c)\, f(a,b) = f(a, bc)\, f(b,c).$$

$$\rightsquigarrow \quad 1 \longrightarrow A \longrightarrow E \longrightarrow \pi \longrightarrow 1$$

where $E = \pi \times A$ with group structure:

$$(a, x)\cdot(b, y) := (ab,\ xy f(a,b))$$

associativity $\Longleftrightarrow$ $\delta f = 0$ (Ex).

Note (i) $1_E = (1_\pi,\ f(1_\pi, 1_\pi)^{-1})$

$$(\because (a,x)\cdot(1, f(1,1)^{-1}) = (a\cdot 1,\ x\cdot f(1,1)^{-1}\overbrace{f(a,1)}^{=f(1,1)\ (\because\ \delta f = 0)}) = (a, x)\ )$$

(ii) $(a, x)^{-1} = (a^{-1},\ x^{-1} f(a, a^{-1})^{-1})$ (Ex).

(iii) $1 \longrightarrow A \longrightarrow E \longrightarrow \pi \longrightarrow 1$ central extension.

$$(a, x) \longmapsto a$$

▨

Eg. $\pi = \mathbb{Z}_2$

$B\pi = K(\pi, 1) = \mathbb{RP}^\infty$

$H^\bullet(\mathbb{RP}^\infty, \mathbb{Z}_2) = \mathbb{Z}_2[x] \Rightarrow H^1 \simeq \mathbb{Z}_2 \simeq H^2$

Indeed, $H^1(\mathbb{Z}_2 ; \mathbb{Z}_2) = \mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$

$H^2(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$, i.e. $\exists$ 2 central ext^n of $\mathbb{Z}_2$ by $\mathbb{Z}_2$, namely

$0 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to \mathbb{Z}_2 \to 0$ and $0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$.

Recall $K(\pi, 1) = |B\pi|$. More generally $\forall\, q \geq 1$

$K(\pi, q) = |Z^q_\pi|$, where

$$Z^q_\pi : ((\mathrm{Ord})) \longrightarrow ((\mathrm{Set})) \quad \text{cosimplicial set}$$
$$[n] \longmapsto Z^q_\pi([n], \pi) = \{q\text{-cycles}\}$$

Remark: Given any

simplicial space $\tilde{X} : ((\mathrm{ord})) \xrightarrow{\text{covariant}} ((\mathrm{Top}))$ $\left(\begin{smallmatrix}\text{e.g.}\\ \tilde{X}=\Delta\end{smallmatrix}\right)$,

and co-simplicial space $\tilde{Y} : ((\mathrm{ord})) \xrightarrow{\text{contravariant}} ((\mathrm{Top}))$,

we can glue them together

$$\tilde{X} \times \tilde{Y} := \coprod_n X[n] \times Y[n] / \sim .$$

When $\tilde{X} = \Delta$, this is just the realization,

i.e. $\Delta \times \tilde{Y} = |\tilde{Y}|$.

Remark: $\forall$ Lie group G,

$\mathcal{N}G$ is a simplicial manifold and

$|\mathcal{N}G| = B_G$, classifying space for G-bundles.

Eg. of cosimplicial space, Čech theory.

X manifold w/ open covering $\mathcal{U}$. $\rightsquigarrow$ cosimp. space

$$X^{\mathcal{U}}: \Big(X \underset{\text{NOT part of } X^{\mathcal{U}}}{\not\longleftarrow}\Big) \amalg U_\alpha \Leftarrow \amalg \overbrace{U_\alpha \cap U_\beta}^{U_{\alpha\beta}} \Lleftarrow \amalg U_{\alpha\beta\gamma} \cdots\cdots$$

$$\begin{tikzcd} ((\mathrm{Ord})) \arrow[r, "X^{\mathcal{U}}", "\text{contra}"'] \arrow[rr, bend right, "\text{(covariant) simplicial dga}"'] & ((\mathrm{Top})) \arrow[r, "\Omega^*", "\text{contra}"'] & ((\mathrm{DGA})) \end{tikzcd}$$

$\xrightarrow[\delta = \Sigma(\pm 1)]{\sim\sim}$ cpx. of DGA

i.e. $$\bigoplus_\alpha \Omega^*(U_\alpha) \xrightarrow{\delta} \bigoplus_{\alpha,\beta} \Omega^*(U_{\alpha\beta}) \xrightarrow{\delta} \bigoplus_{\alpha,\beta,\gamma} \Omega^*(U_{\alpha\beta\gamma}) \cdots\cdots$$
$$\downarrow d \qquad\qquad\qquad \downarrow d$$

double complex (d, δ) $\rightsquigarrow$ $D = \delta \pm d$.

$$\check{H}^\bullet_{dR}(X; \mathbb{R})_{\mathcal{U}} := H^\bullet_D(\text{above complex})$$

$$\check{H}^\bullet_{dR}(X; \mathbb{R}) := \varinjlim_{\mathcal{U}} \check{H}^\bullet_{dR}(X, \mathbb{R})_{\mathcal{U}}$$

Čech-deRham cohomology.

Remark: Čech cohomology.

$\mathcal{U}$ good cover $\Rightarrow$ $\mathcal{N}erve(\mathcal{U})$ cosimplicial set

$\Longrightarrow$ $\mathbb{R}(\mathcal{N}erve(\mathcal{U}))$ simplicial group

$\overset{\delta}{\Longrightarrow}$ complex $\overset{H^\cdot}{\Longrightarrow}$ Čech cohomology.

Remark: Cohomology w/ non-Abelian coeff G.

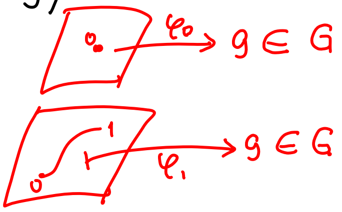
$C^0(M, G) \ni$

$\delta \downarrow$

$C^1(M, G) \ni$

$\delta \downarrow$

$C^2(M, G)$



$\downarrow \delta$ $\quad \delta\varphi_0(01) = \varphi_0(1)^{-1}\varphi_0(0) \in G$

(can also use $\varphi_0(0)\varphi_0(1)^{-1}$).

$$\delta\varphi_1(\langle 012 \rangle) = \varphi_1(01)\,\varphi_1(02)^{-1}\varphi_1(12)$$

$\forall\, \varphi_0 \in C^0, \delta^2\varphi_0 = 1$. But NOT group homo $\Rightarrow$ $C^1/\delta C^0$ makes no sense.

But $C^0 \curvearrowright C^1$ (~ gauge transf.) $\varphi_1^{\varphi_0}(01) := \varphi_0(1)^{-1}\varphi_1(01)\varphi_0(0)$

$$H^1(M, G) \triangleq Z^1(M, G) / C^0(M, G) \quad \text{(action as above.)}$$

§ Characteristic classes.

Principal G-bundle $\begin{matrix} G \to P \\ \downarrow \\ M \end{matrix}$ $\quad P|_{U_\alpha} = U_\alpha \times G$ w/ $g_{\alpha\beta}: U_{\alpha\beta} \to G$ gluing functions.

$M \supset U_\alpha$, $\quad \mathcal{U} = \{U_\alpha\}$: trivializing cover of M.

G ↗ connected.

$$\begin{tikzcd}
G \arrow[d] & \rightsquigarrow & ((\mathrm{Ord})) \arrow[rr, "NG"] & {} & ((\mathrm{Top})): & \mathrm{pt.} & G \arrow[l, shift left] \arrow[l, shift right] & G\times G \arrow[l] \arrow[l, shift left=2] \arrow[l, shift right=2] & \cdots \\
P \arrow[d] & \rightsquigarrow & ((\mathrm{Ord})) \arrow[rr, "M_{\mathcal{U}}"] & {} \arrow[u, "\text{morphism of functors}"'] & ((\mathrm{Top})): & \coprod_\alpha U_\alpha \arrow[u] & \coprod_{\alpha\beta} U_{\alpha\beta} \arrow[l, shift left] \arrow[l, shift right] \arrow[u, "\begin{smallmatrix} g_{\alpha\beta}(x) \\ \uparrow \\ x \end{smallmatrix}"'] & \coprod_{\alpha,\beta,\gamma} U_{\alpha\beta\gamma} \arrow[l] \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[u, "{(g_{\alpha\beta}, g_{\beta\gamma})}"'] & \cdots \\
M & & & & & & & &
\end{tikzcd}$$

$$\xRightarrow{\text{apply } \Omega^\bullet} \quad \Omega^\bullet_{dR}(NG) \longrightarrow \Omega^\bullet_{dR}(M_{\mathcal{U}})$$

$$\rightsquigarrow \quad F: H^\bullet_{dR}(NG) \longrightarrow H^\bullet(M)$$

this gives characteristic classes of P/M.

Eg. $G = \mathbb{C}^\times$

$$\begin{tikzcd}
 & & \Omega^2(\mathbb{C}^\times) & \cdots \\
 & & & \vdots \\
\Omega^\bullet(NG): & \mathbb{C} \arrow[r] & \Omega^1(\mathbb{C}^\times) \arrow[uu, "d"'] \arrow[r, "\delta"] & \Omega^1(\mathbb{C}^\times \times \mathbb{C}^\times) \cdots \\
 & & \varphi \arrow[u, phantom, sloped, "\in"] &
\end{tikzcd}$$

$$D\varphi = 0 \iff d\varphi = 0 \text{ and } \mu^*\varphi = \varphi \otimes 1 + 1 \otimes \varphi$$

where $\mu: \mathbb{C}^\times \times \mathbb{C}^\times \xrightarrow{\text{multi}} \mathbb{C}^\times$

say $\varphi = \frac{1}{2\pi i}\frac{dz}{z}$ ✓ (i.e. $\frac{d(zw)}{zw} = \frac{dz}{z} + \frac{dw}{w}$).

<u>Claim</u>: $H^\bullet(\Omega^\bullet(N\mathbb{C}^\times)) = \mathbb{C}[c_1]$ w/ $c_1 := [\varphi]$

Pf: $E_2^{p,q} = H_\delta(H_d(\Omega^\bullet(NG))) \Longrightarrow H^\bullet(\Omega^\bullet(NG))$

E_1:

$$\begin{array}{|llll}
 & & & \\
0 & H^1\mathbb{C}^\times & H^1(\mathbb{C}^\times\times\mathbb{C}^\times) & \cdots \\
\mathbb{C} & H^0\mathbb{C}^\times & H^0(\mathbb{C}^\times\times\mathbb{C}^\times) & \cdots \\
\hline
\end{array}$$

i.e. $\mathbb{C} \xrightarrow{\delta} H^1(S^1) \xrightarrow{\delta} H^1(S^1) \otimes H^1(S^1) \xrightarrow{\delta} \cdots \rightsquigarrow \mathcal{T}H^\bullet(S^1) = \mathcal{T}\Lambda[\varphi]$

tensor alg. ↙

$\delta\varphi = \varphi \otimes 1 + 1 \otimes \varphi$

$\Longrightarrow H_\delta(\mathcal{T}\Lambda[\varphi]) = S^\bullet[\varphi] = \mathbb{C}[c_1]$

▨

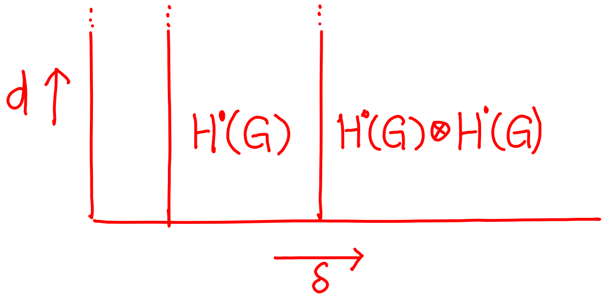
∀ Lie group G $\qquad G\times G \xrightarrow[\text{multi.}]{\mu} G$

$\rightsquigarrow \mu^*: H^\bullet(G) \longrightarrow H^\bullet(G)\otimes H^\bullet(G)$

$\rightsquigarrow (H^\bullet(G), \cup, \mu^*)$ Hopf alg.

$$\begin{tikzcd}
EG \arrow[d] & : & G \arrow[d] & G\times G \arrow[l, shift left] \arrow[l, shift right] \arrow[d] & G\times G\times G \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \arrow[d] & \cdots \\
BG & : & pt & G \arrow[l, shift left] \arrow[l, shift right] & G\times G \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] & \cdots
\end{tikzcd}$$

$$E_2 = H_\delta H_d(\Omega^\bullet(NG)) \Longrightarrow H_D^\bullet(\Omega^\bullet(NG))$$



$\delta: H^\bullet(G) \longrightarrow H^\bullet(G)\otimes H^\bullet(G)$ is induced by

$$\begin{tikzcd}
G & G\times G \arrow[l, shift right=3, "\pi_2"'] \arrow[l, "\mu"'] \arrow[l, shift left=3, "\pi_1"']
\end{tikzcd}$$

$\pi_2(a,b) = b$
$\mu(a,b) = ab.$
$\pi_1(a,b) = a$

$$\delta = \pi_2^* - \mu^* + \pi_1^*$$

$\pi_2^*\varphi = 1\otimes\varphi$
$\pi_1^*\varphi = \varphi\otimes 1$
$\mu^*\varphi$?

Write $\mu^*\varphi = 1\otimes A + C + B\otimes 1$ w/ $C \in H^{>0}\otimes H^{>0}$

· $G \xrightarrow{l_e} G\times G \xrightarrow{\mu} G$, $\mu\circ l_e = 1 \Rightarrow B=\varphi$
$g \mapsto (g,e)$

· Similarly, $A=\varphi$

$\Rightarrow \delta H^\bullet(G) \subseteq H^{>0}(G)\otimes H^{>0}(G).$

(Same for all Hopf alg.).

Def φ primitive $\overset{\Delta}{\Longleftrightarrow} \delta\varphi = 0 \Longleftrightarrow \mu^*\varphi = 1\otimes\varphi + \varphi\otimes 1$

$\forall$ Lie group G

$H^\bullet(G) \supseteq \mathcal{P}H^\bullet(G) \triangleq \{\text{primitives}\}$

Prop: $E_2 = H_\delta H_d(\Omega^\bullet(NG)) = S^\bullet \mathcal{P}H^\bullet(G)$

$d_2 = 0$

Cor.: $H^\bullet(\Omega^\bullet NG) = S^\bullet \mathcal{P}H^\bullet(G)$.

Theorem (Hopf) $H^\bullet(G) = \Lambda^\bullet \mathcal{P}H^\bullet(G)$

More generally,

Theorem (Hopf) $\forall$ H-space (i.e. group axioms up to homotopy)

$H^\bullet(X,\mathbb{R}) = \Lambda(x_1,\dots,x_\ell) \otimes S^\bullet(u_1,\dots,u_s)$ w/ $\deg x_\cdot$ odd, $\deg u_\cdot$ even.

Eg. $H^\bullet(\Omega S^2) = \Lambda(x_1) \otimes S(u_2)$

$H^\bullet(\Omega S^3) = S(u_2)$.

Idea of proof of Hopf thm. ($\Rightarrow$ S^2 NOT topo. gp.).

If $\mu : S^2 \times S^2 \longrightarrow S^2$ group $x \in H^2(S^2)$

$\exists$ 1 $\Longrightarrow$ $\mu^* x = x \otimes 1 + 1 \otimes x$ (x : generator of lowest deg.)

$\Longrightarrow$ $\mu^*(x^2) = x^2 \otimes 1 + 2\, x \otimes x + 1 \otimes x^2$

$x^2 = 0$ $\Rightarrow$ $x \otimes x = 0$ $(-\!\!\times\!\!-)$

(For S^{odd} $\rightsquigarrow$ $x \otimes x + (-1)\, x \otimes x = 0$ no contradiction).

§ de Rham perspective.

G compact connected Lie group.

Use left translation $G_L \curvearrowright \Omega^\bullet(G)$

$$\Omega^\bullet(\mathfrak{g}) := (\Omega^\bullet(G))^{G_L} \quad (\simeq \Lambda^\bullet \mathfrak{g}^*) \circlearrowright d$$

Lie alg. cohomology

$$H^\bullet(\mathfrak{g}, \mathbb{R}) := H^\bullet_d(\Omega^\bullet(\mathfrak{g}))$$

- $$\begin{tikzcd} \Omega^\bullet(\mathfrak{g}) \arrow[r, hook, shift left] & \Omega^\bullet(G) \arrow[l, two heads, shift left, "\int"] \end{tikzcd}$$ (∵ G compact) (i.e. $\omega \mapsto \int_G L_g\omega \, dg$)

- G Connected $\Longrightarrow L_g \overset{\text{h.e.}}{\sim} 1$

$$\xRightarrow{\text{Hodge theory}} \quad H^\bullet(\mathfrak{g}) = H^\bullet_{dR}(G)$$

Use both left and right translation,

$$H^\bullet(G) = H^\bullet(\Omega^\bullet(G)^{G_L \times G_R})$$

$$= H^\bullet(\Omega^\bullet(\mathfrak{g})^{\mathrm{Ad}G})$$

$$\overset{\text{claim}}{=} (\Omega^\bullet \mathfrak{g})^{\mathrm{Ad}G}$$

Pf. of claim: On $(\Omega^\bullet \mathfrak{g})^{\mathrm{Ad}G}$, $d = 0$.

$\iota : G \xrightarrow{\text{inverse}} G$, $\iota(g) = g^{-1}$

$\rightsquigarrow \iota^* : (\Omega^\bullet \mathfrak{g})^{\mathrm{Ad}G} \circlearrowright$

$\iota^*|_{\mathfrak{g}^*} = (-1) \Longrightarrow \iota^*|_{\Lambda^q \mathfrak{g}^*} = (-1)^q$

$[d, \iota^*] = 0 \Rightarrow d = -d \Rightarrow d = 0$

G compact connected Lie group

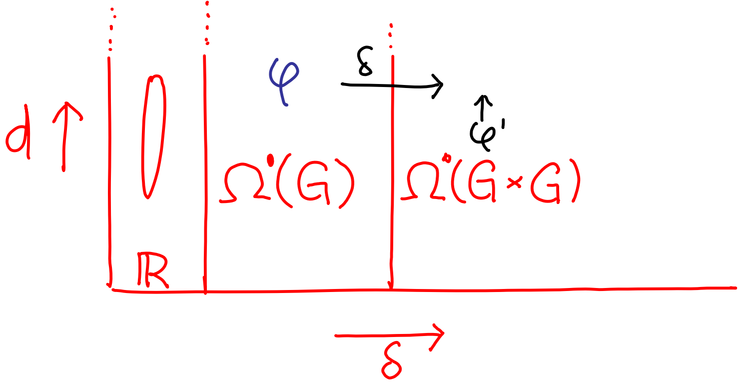
Recall $E_2 = H_d H_\delta(\Omega^\bullet(NG)) \Longrightarrow H_D^\bullet(\Omega^\bullet(NG))$

Thm. $H^p_\delta(\Omega^q(NG)) \simeq \begin{cases} 0 & p\neq q \\ (S^q\mathfrak{g}^*)^{AdG} & p=q \end{cases}$

Cor. $E_1 = E_\infty$, i.e.

$$H^\bullet(BG) \simeq (S^\bullet\mathfrak{g}^*)^{AdG}$$

$\forall$ primitive $[\varphi]\in H^{2n-1}(G)$



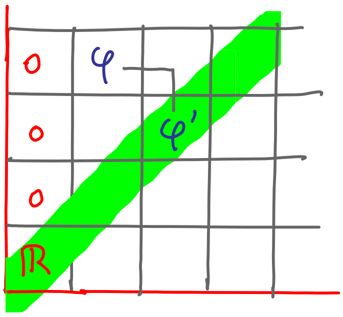
E_1-page.

$[\varphi]$ primitive $\Longleftrightarrow \delta[\varphi]=0$

$\Longleftrightarrow \delta\varphi = d\varphi'$ $\exists\varphi'$

$\delta\varphi' \overset{?}{=} 0$

Claim



φ can be completed to a cocycle in $\Omega^\bullet(NG)$, which extends no further than diagonal.

Eg. $\varphi\in H^3(SU(2)) \rightsquigarrow \delta\varphi = d\varphi'$, $d\varphi'=0$

$\rightsquigarrow [\varphi\pm\varphi'] = p_1 \in H^4(BSU(2))$

§ Noncompact Lie group.

G noncompact. Then $H^\bullet(G) \neq H^\bullet(\mathfrak{g})$

Eg $G = SL(2,\mathbb{R}) \curvearrowright \mathbb{H}$ transitive

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

$\mathrm{Stab}(i) = SO(2)$

$\Longrightarrow SL(2,\mathbb{R})/SO(2) = \mathbb{H}$ contractible

i.e. $SO(2) \xhookrightarrow{\text{h.e.}} SL(2,\mathbb{R})$

$$\Longrightarrow H^\bullet(SL(2,\mathbb{R})) = H^\bullet(\underbrace{SO(2)}_{S^1}) = \Lambda(x) \quad \deg x = 1$$

$$\nparallel$$

$$H^\bullet(\mathfrak{sl}(2,\mathbb{R})) = H^\bullet(\mathfrak{su}(2)) \quad (\because \text{same complexification})$$

$$= H^\bullet(SU(2)) \quad (\because SU(2) \text{ cpt. Lie gp})$$

$$= \Lambda(y) \quad \text{w/} \quad \deg y = 3.$$

In general, $G > K$ max. cpt. subgroup (unique up to conjugation)

G/K contractible, in particular,

$$\Rightarrow H^\bullet(G) = H^\bullet(K)$$

Def: Continuous cohomology

$$NG : G/G \leftleftarrows G\times G/G \mathrel{\substack{\leftarrow\\\leftarrow\\\leftarrow}} G\times G\times G/G \ \cdots\cdots$$

$$\Omega^0(NG) : \mathbb{R} \longrightarrow \Omega^0(G) \xrightarrow{\delta} \Omega^0(G\times G) \xrightarrow{\delta} \cdots\cdots$$

$$H^\bullet_{cts}(G) := H^\bullet_\delta(\Omega^0(NG))$$

Theorem (VanEst)

$\exists$ s.s. $E_2 = H^\bullet(G) \otimes H^\bullet_{cts}(G) \Longrightarrow H^\bullet(\mathfrak{g})$

Theorem. $H^\bullet_{cts}(G) = \mathbb{R}$ if G compact

Theorem. $H^*_{cts}(\mathbb{R}) = \begin{cases} \mathbb{R} & * = 0, 1 \\ 0 & \text{otherwise} \end{cases}$

$H^\bullet_{cts}(\mathbb{R}^n) = \Lambda[x_1, \dots, x_n]$ w/ $\deg x_i = 1$

Theorem. $H^*_{cts}(SL(2,\mathbb{R})) = \begin{cases} \mathbb{R} & * = 0, 2 \\ 0 & \text{otherwise} \end{cases}$

Eg.

$$\begin{tikzcd} H^\bullet(SL(2,\mathbb{R})) & \mathbb{R} \arrow[rrd, "d_2", "\cong"'] & 0 & \mathbb{R} & & 0 & 0 & \mathbb{R} & = H^\bullet(\mathfrak{sl}(2,\mathbb{R})) \\ & \mathbb{R} & 0 & \mathbb{R} & \Longrightarrow & \mathbb{R} & 0 & 0 & \\ & & H^\bullet_{cts}(SL(2,\mathbb{R})) & & & & & & \end{tikzcd}$$

* * *

$H^\bullet(BG)$ can be computed via $* \sim EG \overset{\text{free}}{\curvearrowleft} G$.
Want to replace by "smaller" model algebraically.
(free action ⟿ injective module).

$$\Omega^0(NG) = \Omega^0(EG)^G \quad (\text{Not true for } \Omega^{>0})$$

Now assume G discrete ($\Rightarrow \Omega^{>0}(NG) = 0$)

$$H^\bullet(BG) = H^\bullet_\delta(\Omega^0(NG)) = H^\bullet_\delta(\Omega^0(EG)^G)$$

Claim: $\Omega^0(EG)$ is injective resolution of $\mathbb{R}$.

Recall. I, P : G-mod

Def: P projective if

$$\begin{tikzcd} & & P \arrow[d] \arrow[dll, dashed, "\exists"'] & \\ \forall\, A \arrow[rr] & & B \arrow[r] & 0 \end{tikzcd}$$

Def: I injective if

(dual notions)

$$\begin{tikzcd} & & P^* = I & \\ \forall\, A^* \arrow[urr, dashed, "\exists"] & & B^* \arrow[ll] \arrow[u] & 0 \arrow[l] \end{tikzcd}$$

Eg. G discrete group

$$\Omega^0(G) \supset \Omega^0(G)_{fin} = \left\{ \begin{array}{l} \text{functions on } G \\ \text{w/ finite support} \end{array} \right\}$$

$\Omega^0(G)_{fin}$ is projective as we lift δ-functions arbitrarily first and then extends.

$\Rightarrow$ $\Omega^0(G) = (\Omega^0(G)_{fin})^*$ is injective

$\Rightarrow$ $\Omega^0(EG)$ is injective resolution of $\mathbb{R}$ (as G-mod).

Fundamental theorem in Homological Algebra.

$0 \longrightarrow M \longrightarrow I^\bullet$ inj. resol^n of G-mod

$\Rightarrow$ $H^\bullet((I^\bullet)^G)$ indep. of inj. resol^n chosen

$\rotatebox{90}{$\triangleq$}$

$H^\bullet_{EM}(G, M)$.

Prop: Finite group $G \curvearrowright M$, $char(F) = 0$ (or $char F > |G|$)

$$\Rightarrow H^\bullet_{EM}(G, M) = \begin{cases} Inv M = M^G & \bullet = 0 \\ 0 & \text{otherwise} \end{cases}$$

Pf $\because$ resol^n $\Rightarrow$ $\exists$ homotopy operator contracts $I^\bullet$ to M. But not necessary G-equivar.

$|G| < \infty$ $\xrightarrow[\frac{1}{|G|}\int_G]{\text{average}}$ G-equivar. one $\Rightarrow$ $H^{>0} = 0$.

✓

Can compute $H^\bullet(NG)$ using any inj. resol^n of $\mathbb{R}$.

It works for compact G (averaging ✓).

Prop. compact Lie group $G \curvearrowright V$

$$H^\bullet_{cts}(G,V) = \begin{cases} V^G & \bullet = 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$0 \to \mathbb{R} \to \Omega^0(G) \xrightarrow{d} \Omega^1(G) \xrightarrow{d} \cdots$$

resol^n, not injective ($H^\bullet_{dR}(G) \neq 0$).

When $G = \mathbb{R}^n$, it is morally injective. So

$$\begin{aligned} H^\bullet_{cts}(\mathbb{R}^n) &= H^\bullet(\Omega^\bullet(\mathbb{R}^n)^{Inv}) \\ &= \Omega^\bullet(\mathbb{R}^n)^{Inv} \quad (\because d = 0 \text{ on const. forms}) \\ &= \Lambda^\bullet(\mathbb{R}^n)^*. \end{aligned}$$

Hence,

Prop. $H^\bullet_{cts}(\mathbb{R}^n) = H^\bullet(T^n)$.

$\forall\ G \geqslant K$ max. cpt. subgp.,

$0 \to \underline{\mathbb{R}} \longrightarrow \Omega^\bullet(G/K)$ is "injective" resolution

$(\because G/K \overset{h.e.}{\simeq} pt)$

Theorem. $H^\bullet_{cts}(G) = H^\bullet(\Omega^\bullet(G/K)^G)$

$= \Omega^\bullet(G/K)^G$ if G semi-simple

Eg. $SL(2,\mathbb{R})/SO(2) = \mathbb{H}$

$$\Omega^2(\mathbb{H})^{SL(2,\mathbb{R})} = \begin{cases} \mathbb{R}\omega, & \bullet = 2 \quad (\omega \text{ inv. vol. form}) \\ 0, & \bullet = 1 \\ \mathbb{R}1, & \bullet = 0 \end{cases}$$

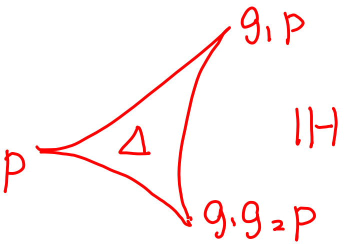
So, $H^2_{cts}(SL(2,\mathbb{R})) \simeq \Omega^2(\mathbb{H})^{SL(2,\mathbb{R})} = \mathbb{R}\langle\omega\rangle$

What is the corresp. repr. $\varphi : G \times G \longrightarrow \mathbb{R}$?

Choose any $p \in \mathbb{H}$, define

$$\varphi_p : G \times G \longrightarrow \mathbb{R}$$

$$\varphi_p(g_1, g_2) = \int_\Delta \omega$$

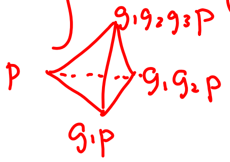


Claim: $\delta \varphi_p = 0$.

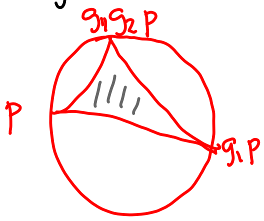
$$\delta\varphi_p(g_1, g_2, g_3)$$

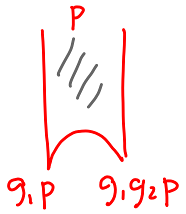
$$= \varphi(g_2, g_3) - \varphi(g_1 g_2, g_3) + \varphi(g_1, g_2 g_3) - \varphi(g_1, g_2)$$

$$\overset{\text{Stokes}}{=} \int (d\omega) \overset{d\omega = 0}{=} 0$$



- $[\varphi_p] \in H^2_\delta(\Omega^0(NG))$ is indep of $p \in \mathbb{H}$.
- If we move $p \in \mathbb{H}$ to $p_\infty \in \partial\mathbb{H} = S^1$





Area $\in \pi\mathbb{Z} \subset \mathbb{R}$

$\Rightarrow \frac{1}{\pi}\varphi_{p_\infty}$ is $\mathbb{Z}$-valued cocycle.

For general, $\omega \in \Omega^n(G/K)^G$, $(d\omega = 0)$, then

$$\varphi_p : \prod^n G \longrightarrow \mathbb{R}$$

$\varphi_p(g_1, \dots, g_n) = \int_\Delta \omega$ w/ $\Delta = \langle p, g_1 p, g_1 g_2 p, \dots \rangle \subset G/K$

gives a cocycle in $H^n_\delta(\Omega^0(NG))$.

$G = GL(n, \mathbb{R})$

$\Theta = x^{-1} dx \in \Omega^1(G, \mathfrak{g})^{GL}$

$d\Theta + \Theta \wedge \Theta$ structure eqt. ($\Leftrightarrow$ Jacobi id.)

$G = SL(2, \mathbb{R})$, $\Theta = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix}$, $\theta_{11} + \theta_{22} = 0$

$\vee\!/$

$K = SO(2) \simeq S^1$, $\theta = (\theta_{12} - \theta_{21})/2$, generate $H^1(S^1)$.

str. eqt. $\Longrightarrow$ $d\theta + \alpha \wedge \varphi = 0$

w/ $\varphi = \frac{\theta_{12} + \theta_{21}}{2}$; $\alpha = \frac{\theta_{11} - \theta_{22}}{2} = \theta_{11}$.

Claim: $d\theta$ is basic form wrt $K \curvearrowright G$.

i.e. $$\begin{tikzcd} S^1 = K \arrow[r] & G \arrow[d, "\pi"] \\ & G/K \end{tikzcd}$$ $d\theta = -\pi^* \omega$ $\exists\, \omega \in \Omega^2(G/K)$

(i.e. ω = curv. of S^1-conn. $\operatorname{Ker} \theta$.)

Let $X = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ (left inv.) vector field, generates S^1-action.

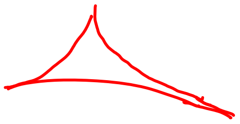
$\iota_X \theta = 1$, $\iota_X \alpha = 0 = \iota_X \varphi$ ✓

$\Longrightarrow$ $\iota_X d\theta = \iota_X (\varphi \wedge \alpha) = 0$

$\mathcal{L}_X d\theta = 0$ ✓ $\Big\} \Rightarrow d\theta$ basic.

Claim: ω = area form.

i.e. $\forall$ geodesic triangle $T \subset \mathbb{H}$

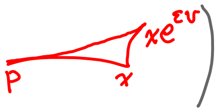


$$\int_T \omega \overset{?}{=} \left(\sum \text{exterior angles} - 2\pi\right)/2\pi$$

$$\left(\int_T \omega = \int_{\tilde{T}} \overbrace{\pi^* \omega}^{-d\theta/2\pi} \overset{\text{Stokes'}}{=\!=} \frac{1}{2\pi} \int_{\partial \tilde{T}} \theta \quad \text{choose suitable lift} \Rightarrow ✓ \right)$$

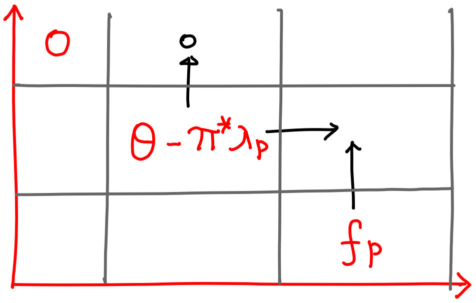
↑ lift to G

Integrate from $p \in \mathbb{H} \rightsquigarrow \omega = -d\lambda_p$ $\exists \lambda_p \in \Omega^1(\mathbb{H})$.

More precisely, $\lambda_p(x)(v) \triangleq \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\langle p, x, xe^{\varepsilon v} \rangle} \omega$ ()

So $\quad d(\theta - \pi^* \lambda_p) = 0$

Claim: $\delta(\theta - \pi^* \lambda_p) = df_p$.



$\Omega^\bullet(NG)$.

Namely, $(0, \theta - \pi^* \lambda_p, f_p) \in H^\bullet(\Omega^\bullet(NG)) = H^\bullet(BG)$ represents first Chern class c_1 for $SL(2, \mathbb{R})$-bdl.